Density of Binary Disc Packings

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Sphere packings

**Sphere packing**: interior disjoint unit spheres.

**Density**: limsup of the proportion of $B(0, r)$ covered.

**Questions**: maximum density? densest packings?
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The maximum density of sphere packings in $\mathbb{R}^2$ is $\frac{\pi}{2\sqrt{3}} \approx 0.9069$. 
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![Sphere packing diagram]
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*The maximum density of sphere packings in $\mathbb{R}^3$ is $\frac{\pi}{3\sqrt{2}} \approx 0.7404$.***
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Theorem (Vyazovska, 2017)

The maximum density of sphere packings in $\mathbb{R}^8$ is $\frac{\pi^4}{384} \approx 0.2536$.

It is reached for spheres centered on the $E_8$ lattice.
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Theorem (Vyazovska et al., 2017)

The maximum density of sphere packings in $\mathbb{R}^{24}$ is $\frac{\pi^{12}}{12!} \approx 0.0019$.

It is reached for spheres centered on the Leech lattice.
Unequal sphere packings

The density becomes parametrized by the ratios of sphere sizes. Natural problem in materials science!
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Simplest non-trivial case: two discs in $\mathbb{R}^2$, i.e., binary disc packings.
The maximum density is a function $\delta(r)$ of the ratio $r \in (0, 1)$.
The hexagonal compact packing yields a uniform lower bound.
Lower bounds

Any given packing yields a lower bound for a specific $r$. 
Lower bounds

\[ \delta(r) \]

It can be extended over a neighborhood of \( r \) (more or less cleverly).
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Upper bounds

First upper bound by Florian in 1960.
Blind’s bound is tight for $r \geq \sqrt{\frac{7\tan(\pi/7) - 6\tan(\pi/6)}{6\tan(\pi/6) - 5\tan(\pi/5)}} \approx 0.743$
On the other side: \( \lim_{r \to 0} \delta(r) = \frac{\pi}{2\sqrt{3}} + \left(1 - \frac{\pi}{2\sqrt{3}}\right) \frac{\pi}{2\sqrt{3}} \approx 0.9913 \)
Tight bounds

The exact maximum density is also known for 9 "magic" ratios!
Theorem (Heppes’00, Heppes’03, Kennedy’04, Bédaride-F.’20)

These periodic binary disc packings have maximum density.
Compact packings

Theorem (Kennedy, 2006)

The ratios are those that allow for a triangulated contact graph.
The disc ratio of a compact packing is determined by the contacts.
Allowing some discs to separate may give a degree of freedom...
Flipping and flowing

...that can be used to vary continuously the ratio...
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...that can be used to vary continuously the ratio...
Flipping and flowing... until it is blocked by new contacts.
Flipping and flowing

Some cases may be tricky: how many (which) contacts to keep?
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Lower bounds reloaded

Flipping and flowing greatly improves the lower bound.
Lower bounds reloaded

Flipping and flowing greatly improves the lower bound. Is it tight?
Other dense packings

For small ratio, there are many dense packings.
But they seem to become more sparse as the ratio grows.
Phase separation

Can we at least do better than the hexagonal compact packing?
Theorem (F., to be improved)

For $r \in [0.445, 0.514] \cup [0.566, 0.627] \cup [0.647, 1)$, $\delta(r) = \frac{\pi}{2\sqrt{3}}$. 
Binary and ternary superlattices self-assembled from colloidal nanodisks and nanorods.

T. Paik, B. Diroll, C. Kagan, Ch. Murray  
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Infinite-pressure phase diagram of binary mixtures of (non)additive hard disks.

Phase diagram

Based on intensive Monte-Carlo simulations;

The concept of "phase" needs to be formalized.
Equivalent disc packings

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Theorem (F. 2020)

*The densest disc packings with a proportion $x$ of large discs are:*

- twinnings of *two periodic packings* for $x \leq 0.5$;
Playing with stoichiometry for $r = \sqrt{2} - 1$

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Summary & perspectives

- Compact packings are good at maximizing the density;

- Flipping and flowing provides a good way to limit density loss;

- Phase separation (hexagonal packing) is almost characterized.

- Still lot of work to get the whole phase diagram.

- Maximum density enforced by attractive forces?

- More discs?

D. Pchelina's lightning talk for 3 discs.

- Higher dimensions (e.g., rock salt)?
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The overall density is bounded by checking inequalities over a **compact** set of triangles using **computer interval arithmetic**.